HYDRODYNAMIC COUPLED MODES OF A NEMATIC UNDER A TEMPERATURE GRADIENT AND A UNIFORM GRAVITATIONAL FIELD

J. Camacho[1] and R. F. Rodríguez[2,3,4].
[1]Maestría en Ciencias de la Complejidad,
Universidad Autónoma de la Ciudad de México.
[2]Instituto de Física, Universidad Nacional Autónoma de México.
[3]Correspondence author. E-mail: zepeda@fisica.unam.mx.
[4]Fellow of SNI, México.

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Abstract

Fluctuating hydrodynamics (FH) describes the dynamics of the fluctuations for fluids at mesoscopic scales. Here we use this approach to study the fluctuations of the hydrodynamic variables of a thermotropic nematic liquid crystal (NLC) in a nonequilibrium steady state $(NESS)$. This state is induced by an externally imposed temperature gradient and a uniform gravity field. We calculate analytically both, the equilibrium and nonequilibrium hydrodynamic modes. We find that in this $NESS$ the nonequilibrium effects produced by the external gradients only affect the longitudinal variables. This gives rise to a pair of sound modes, one orientation mode of the director and two visco-heat modes formed by the coupling of the shear and thermal modes. We also find that the last three modes exhibit the largest changes. The analytical expressions that we have found for the visco-heat modes imply that the heat and shear modes of the NLC are coupled, that they reduce to those of simple fluid in the isotropic limit and that these modes may become propagative, a feature that also occurs in the simple fluid. In the isotropic limit of the nematic our results also reduce to the hydrodynamic modes of a simple fluid in the presence of the same temperature gradient and the pressure gradient produced by the gravity field.

1 INTRODUCTION

The Landau and Lifshitz theory of hydrodynamic fluctuations close to equilibrium [1], was put on a firm basis within the framework of the general theory of stationary Gaussian Markov processes by Fox and Uhlenbeck [2], [3]. In fluctuating hydrodynamics

(FH) the usual deterministic hydrodynamic equations are supplemented with random dissipative fluxes of thermal origin, obeying fluctuation-dissipation relations. This approach has matched the theory of Onsager and Machlup with the approach of Landau and Lifshitz, for systems where the basic state variables do not posses a definite time reversal symmetry, leading to Langevin-like stochastic equations for the evolution of the fluctuations of the state variables [4], [5], [6]. In this way fluctuating hydrodynamics provides a systematic method for assessing the nature of spontaneous fluctuations induced by intrinsic thermal noise.

The Fox and Uhlenbeck's scheme has been applied to simple fluids and their binary mixtures [7], [8], [9]; however, more recently it has been also verified that FH can be extended to deal with thermally excited fluctuations in complex fluids in stationary nonequilibrium states [10], [11], [12], [13]. In spite of the fact that the theory of fluctuations in nonequilibrium fluids was initiated in the late 70's [14], [15], and pursued by many authors [16], [17], still nowadays several questions concerning the nature of hydrodynamic fluctuations in $NESS$ are of current active interest. One of these issues is the long-range character of these fluctuations even far away from instability points. It has been shown theoretically that the existence of the so called generic scale invariance is the origin of the long range nature of the correlation functions [18]. However, in spite of the considerable interest in fluctuations about dissipative steady states of simple fluids during the last two decades, there are few similar studies for equilibrium or nonequilibrium stationary states of complex fluids [19], [20].

The basic purpose of the present paper is to describe the dynamics of the fluctuations of the hydrodynamic variables of a nematic liquid crystals in a $NESS$ induced by a stationary temperature gradient and under the influence of gravity [21]. More specifically, we compare the fluctuations in the presence of a dissipative thermodynamic force like a stationary temperature gradient, with those in the presence of a conservative thermodynamic force like gravity. This comparison has been analyzed for a simple fluid [22], and since gravity is a small force, significant changes in the fluctuations only occur at wave numbers that are too small to be probed experimentally. However, it is important to know theoretically how such small force can affect the fluctuations also in a liquid crystal. To our knowledge, this issue has only been explored for liquid crystals in the Refs. [23], [24] for the same $NESS$ considered in this work. However, we consider that their conclusions are not definitive because in the isotropic limit, they do not reduce to the well known corresponding visco-heat modes of a simple fluid [25]. References [23], [24] predict that the thermal and director diffusive modes are coupled. In contrast, the analytical expressions that we have found for the visco-heat modes imply, on the one hand, that the heat and shear modes of the NLC are coupled and that they reduce to those of simple fluid in the isotropic limit. Furthermore, our expressions also predict that these modes may become propagative, a feature that also occurs in the simple fluid [22].

To this end and on the basis of FH, we first evaluate the equilibrium and nonequilibrium hydrodynamic modes of the NLC and then we show that among the longitudinal modes, which are the only ones affected by the external gradients, there is a pair of visco-heat modes constituted by the coupling of the shear and thermal modes. These modes are more general than those reported so far in the literature for a NLC in a $NESS$ produced by a temperature gradient [23], [24], because they reduce to the corresponding expressions in the isotropic limit of the NLC [26], [27], [28], to those of a simple fluid [9], [22], [25].

2 MODEL

Consider a quiscent thermotropic nematic liquid crystal thin layer of thickness d confined between two parallel plates in a homeotropic arrangement, $\widehat{n}_0 = (0,0,1)$. The NLC is in the presence of a uniform gravitational field $\overrightarrow{g} = -g\widehat{z}$, where $\widehat{z}$ denotes the unitary vector along the z-axis as depicted in Fig. 1. The transverse dimensions of the cell along the x and y directions are large compared to d.

Figure 1: Schematic representation of the homeotropic nematic cell under the influence of a constant gravitational field $\overrightarrow{g}$ and an external, uniform, temperature gradient ∇T.

The plates are maintained at the uniform temperatures $T_1 > T_2$ so that a constant temperature gradient

$$\nabla_z T(z) = -\alpha\widehat{z} \tag{1}$$

is established between them. The presence of the gravitational field induces a constant pressure gradient $\nabla_z p$ given by

$$\nabla_z p(z) = -\rho g\widehat{z}, \tag{2}$$

where ρ is the mass density. Furthermore, if the temperature difference between the plates is of only a few degrees, we may assume that $\nabla_z T$ and $\nabla_z p$ do not generate flows or hydrodynamic instabilities, so that the NLC is in a quiscent $NESS$. This state is described by $\psi^{st} \equiv \left\{v_i^{st}, \rho^{st}(z), s^{st}(z), \widehat{n}^{st}(z)\right\}$, where $v_i^{st} = 0$ is the hydrodynamic velocity and ρ^{st}, s^{st}, $\widehat{n}^{st}$ denote, respectively, the mass, entropy, $s^{st}(z)$ and director

$\widehat{n}^{st}(z)$ local densities of the NLC in the stationary state. The gradient of the state variables ψ^{st} can be expanded in a Taylor series around the equilibrium state (p_0, T_0) in terms of the external gradients. For the boundary conditions $T(z = -d/2) = T_1$, $T(z = d/2) = T_2$ and to first order in these gradients, the stationary temperature profile is then given by

$$T^{st}(z) = T_0\left(1 - \frac{\alpha}{T_0}z\right), \tag{3}$$

where $T_0 \equiv T^{st}(z=0) = (T_1 + T_2)/d$, $\alpha \equiv (T_1 - T_2)/d$ and

$$\nabla_z s^{st} = \frac{c_p}{T_0} X\widehat{z}, \tag{4}$$

$$\nabla_z \rho^{st} = -\rho_0 \beta\left(X + \frac{g\beta T_0}{c_p(\gamma - 1)}\right)\widehat{z}. \tag{5}$$

Note that the effective temperature gradient $X \equiv -\alpha + \frac{g\beta T_0}{c_p}$ contains the contributions of both external gradients (α and g), and c_p, β, c_T are, respectively, the specific heat at constant pressure, the thermal expansion coefficient and the isothermal sound velocity of the nematic. To arrive at Eqs. (4) and (5) we have used the thermodynamic relations $\beta^2 \equiv (\gamma - 1)c_p/T_0 c_s^2$ and $\gamma = c_s^2/c_T^2$, where c_s is the adiabatic sound velocity.

2.1 Fluctuating nematodynamics

The mass conservation equation for ρ, the equation of motion for $\overrightarrow{v}$, the balance equation for s and the relaxation equation for the director field $\widehat{n}$ are given, respectively, by [26], [29], [30], [31], [32],

$$\left(\frac{\partial}{\partial t} + v_j \nabla_j\right)\rho + \rho\nabla_l v_l = 0, \tag{6}$$

$$\rho\left(\frac{\partial}{\partial t} + v_j\nabla_j\right)v_i = -\nabla_i p + \nabla_j \sigma'_{ij} - \nabla_j\left(\Phi_{jl}\nabla_i n_l\right) - \frac{1}{2}\nabla_j\left(\lambda_{imj}h_m\right) + \rho f_i, \tag{7}$$

$$\rho T\left(\frac{\partial s}{\partial t} + v_i \nabla_i s\right) = \sigma'_{ij}\nabla_j v_i - \nabla_j q_j + h_k \mathcal{N}_k \tag{8}$$

$$\frac{\partial n_i}{\partial t} + v_j \nabla_j n_i = \frac{1}{2}\lambda_{ijk}\nabla_j v_k + \mathcal{N}_i, \tag{9}$$

where $p(\overrightarrow{r}, t)$ is the pressure field and the tensor σ'_{ij} denotes the momentum current

$$\sigma'_{ij} \equiv \nu_{ijlm}\nabla_m v_l. \tag{10}$$

The viscous tensor ν_{ijkl} is

$$\begin{aligned}\nu_{ijkl} \equiv{}& \nu_2(\delta_{jl}\delta_{ik} + \delta_{il}\delta_{jk}) + 2(\nu_1 + \nu_2 - 2\nu_3)n_i n_k n_l \\ &+(\nu_3 - \nu_2)(n_j n_l \delta_{ik} + n_j n_k \delta_{il} + n_i n_k \delta_{jl} + n_i n_l \delta_{jk}) \\ &+(\nu_4 - \nu_2)\delta_{ij}\delta_{kl} + (\nu_5 - \nu_4 + \nu_2)(\delta_{ij} n_k n_l + \delta_{kl} n_i n_j)\end{aligned} \tag{11}$$

where the set $\nu_i = \{\nu_1, \nu_2, \nu_3, \gamma_1, \gamma_2\}$ denotes the five nematic viscosity coefficients of a nematic in the notation of Harvard [29]; f_i is the total body force acting on the nematic; Φ_{ki} is given by

$$\Phi_{ki} = K_{ikrj} \nabla_j n_r, \tag{12}$$

where the fourth order tensor K_{ijkl} depends on the elastic constants K_1 (splay), K_2 (twist), K_3 (bend) and it is defined in terms of the Levi-Civitta tensor ϵ_{ijk} by

$$K_{ijkl} = K_1 \delta_{ij} \delta_{kl} + K_2 n_p \epsilon_{pij} n_q \epsilon_{qkl} + K_3 n_j n_l \delta_{ik}. \tag{13}$$

The third order tensor in Eq. (9)

$$\lambda_{kji} \equiv (\lambda - 1) \delta^{\perp}_{kj} n_i + (\lambda + 1) \delta^{\perp}_{ki} n_j, \tag{14}$$

depends on the orientational viscosities through $\lambda \equiv -\gamma_1/\gamma_2$. The vector q_l is the heat flux

$$q_l \equiv -\kappa_{lj} \nabla_j T, \tag{15}$$

where κ_{ij} is the thermal conductivity tensor

$$\kappa_{ij} = \kappa_{\perp} \delta_{ij} + \kappa_a n_i n_j, \tag{16}$$

with anisotropy $\kappa_a \equiv \kappa_{\|} - \kappa_{\perp}$, being $\kappa_{\perp}$ and $\kappa_{\|}$ its perpendicular and parallel components with respect to the director field. $\mathcal{N}_i$ is the quasi-current associated with the director

$$\mathcal{N}_i = \frac{1}{\gamma_1} \delta^{\perp}_{ik} h_k \tag{17}$$

and the molecular field h_i is

$$h_i = \delta^{\perp}_{ir} K_{rjkl} \nabla_j \nabla_l n_k - \delta^{\perp}_{iq} \left(\frac{1}{2} \frac{\partial}{\partial n_q} K_{pjkl} - \frac{\partial}{\partial n_p} K_{qjkl} \right) \nabla_j n_p \nabla_l n_k. \tag{18}$$

It should be stressed that Eqs. (6)-(9) are valid for any motion of the nematic including the hydrodynamic deviations (fluctuations) of the state variables from the $NESS$. If only the linear deviations, $\delta\rho(\overrightarrow{r}, t) = \rho(\overrightarrow{r}, t) - \rho^{st}$, $\delta v_i(\overrightarrow{r}, t) = v_i(\overrightarrow{r}, t)$, $\delta s(\overrightarrow{r}, t) = s(\overrightarrow{r}, t) - s^{st}$, $\delta n_i(\overrightarrow{r}, t) = n_i(\overrightarrow{r}, t) - n_i^{st}$ around the stationary state $v_i^{st} = 0$, $n_i^{st} = cte$, $h_l^{st} = 0$ are considered and $\Phi_{lj}^{st} = 0$, the linearized equations associated with Eqs. (6)-(9) read

$$\frac{\partial}{\partial t} \delta\rho = -\delta v_j \nabla_j \rho^{st} - \rho^{st} \nabla_l \delta v_l, \tag{19}$$

$$\rho^{st} \frac{\partial}{\partial t} \delta v_i = -\nabla_i \delta p - \frac{1}{2} \lambda^{st}_{kji} \nabla_j \delta h_k + \nabla_j \delta \sigma'_{ij} - g \delta_{iz} \delta\rho, \tag{20}$$

$$\rho^{st} T^{st} \left(\frac{\partial}{\partial t} \delta s + \delta v_j \nabla_j s^{st} \right) = -\nabla_l \delta q_l, \tag{21}$$

$$\frac{\partial}{\partial t} \delta n_i = \frac{1}{2} \lambda^{st}_{ijk} \nabla_j \delta v_k + \delta \mathcal{N}_i, \tag{22}$$

with

$$\delta\sigma'_{ij} \equiv \nu^{st}_{ijkl} \nabla_l \delta v_k. \tag{23}$$

$$\delta q_l \equiv -\delta\kappa_{lj} \nabla_j T^{st} - \kappa^{st}_{lj} \nabla_j \delta T, \tag{24}$$

$$\delta\mathcal{N}_i \equiv \frac{1}{\gamma_1} \left(\delta^{\perp}_{ik}\right)^{st} \delta h_k, \tag{25}$$

where

$$\lambda^{st}_{kji} \equiv (\lambda - 1) \left(\delta^{\perp}_{kj}\right)^{st} n^{st}_i + (\lambda + 1) \left(\delta^{\perp}_{ki}\right)^{st} n^{st}_j, \tag{26}$$

$$\delta h_i = \left(\delta^{\perp}_{ir}\right)^{st} K^{st}_{rjkl} \nabla_j \nabla_l \delta n_k, \tag{27}$$

$$\begin{aligned}
\nu^{st}_{ijkl} &\equiv \nu_2(\delta_{jl}\delta_{ik} + \delta_{il}\delta_{jk}) + 2(\nu_1 + \nu_2 - 2\nu_3) n^{st}_i n^{st}_j n^{st}_k n^{st}_l \\
&+(\nu_3 - \nu_2)(n^{st}_j n^{st}_l \delta_{ik} + n^{st}_j n^{st}_k \delta_{il} + n^{st}_i n^{st}_k \delta_{jl} + n^{st}_i n^{st}_l \delta_{jk}) \\
&+(\nu_4 - \nu_2)\delta_{ij}\delta_{kl} + (\nu_5 - \nu_4 + \nu_2)(\delta_{ij} n^{st}_k n^{st}_l + \delta_{kl} n^{st}_i n^{st}_j),
\end{aligned} \tag{28}$$

$$\delta\kappa_{ij} = \kappa_a \left(n_i \delta n_j + \delta n_i n_j\right), \tag{29}$$

$$\kappa^{st}_{ij} = \kappa_{\perp}\delta_{ij} + \kappa_a n^{st}_i n^{st}_j, \tag{30}$$

$$K^{st}_{ijkl} = K_1 \delta_{ij}\delta_{kl} + K_2 n_p \epsilon_{pij} n_q \epsilon_{qkl} + K_3 n_j n_l \delta_{ik}, \tag{31}$$

where we have defined $\left(\delta^{\perp}_{ir}\right)^{st} = \delta_{ir} - n^{st}_i n^{st}_r$. In Eq. (20) the total volumetric force is $f_i = -\delta_{iz} g$.

Following Landau and Lifshitz [1], we now introduce fluctuating components into the momentum current $\sigma'_{ij}(\vec{r}, t)$, the heat flux q_l and the relaxation quasi-current $\mathcal{N}_i$ of the orientation of the nematic. These stochastic components are denoted, respectively, by $\nabla_j \Sigma_{ij}(\vec{r}, t)$, $\pi_i(\vec{r}, t)$, $\Upsilon_i(\vec{r}, t)$, and are chosen as zero averaged stochastic processes

$$\langle \Sigma_{ij}(\vec{r}, t)\rangle = \langle \pi_i(\vec{r}, t)\rangle = \langle \Upsilon_i(\vec{r}, t)\rangle = 0, \tag{32}$$

satisfying fluctuation-dissipation relations (FDR) which have the same form as in equilibrium, but replacing the equilibrium temperature by T^{st}, [26], [33]. These relations are

$$\left\langle \Sigma_{\alpha j}(\vec{r}, t)\Sigma_{\beta l}(\vec{r}\,', t')\right\rangle = 2k_B T^{st} \nu^{st}_{\alpha\beta jl} \delta\left(\vec{r} - \vec{r}\,'\right)\delta\left(t - t'\right), \tag{33}$$

$$\left\langle \pi_i(\vec{r}, t)\pi_j(\vec{r}\,', t')\right\rangle = 2k_B \left(T^{st}\right)^2 \kappa^{st}_{ij} \delta\left(\vec{r} - \vec{r}\,'\right)\delta\left(t - t'\right), \tag{34}$$

$$\left\langle \Upsilon_\mu(\vec{r}, t)\Upsilon_\nu(\vec{r}\,', t')\right\rangle = 2k_B T^{st} \frac{1}{\gamma_1} \left(\delta^{\perp}_{\mu\nu}\right)^{st} \delta\left(\vec{r} - \vec{r}\,'\right)\delta\left(t - t'\right). \tag{35}$$

Here k_B is Boltzmann's constant and T^{st} is given by Eq. (3). Substitution of Eqs. (1)-(5), (23) and (25) into Eqs. (19)-(22), leads to the following first order in the gradients set of linear, fluctuating nematodynamic equations

$$\frac{\partial}{\partial t}\delta\rho = \rho_0 \beta \left[X + \frac{g\beta T_0}{(\gamma - 1)\, c_p}\right] \delta v_z - \rho_0 \left\{1 - \beta \left[X + \frac{g\beta T_0}{(\gamma - 1)\, c_p}\right] z\right\} \nabla_l \delta v_l, \tag{36}$$

$$\rho_0 \left\{1 - \beta \left[X + \frac{g\beta T_0}{(\gamma - 1)\, c_p}\right] z\right\} \frac{\partial}{\partial t}\delta v_i = -\nabla_i \delta p - \frac{1}{2}\lambda^{st}_{kji} \nabla_j \delta h_k$$
$$+\nu^{st}_{ijlm} \nabla_j \partial_m \delta v_l - g\delta_{iz}\delta\rho + \nabla_j \Sigma_{ij}, \tag{37}$$

$$\rho_0 T_0 \left\{1 - \beta \left[X + \frac{g\beta T_0}{(\gamma - 1)\, c_p}\right] z\right\} \left(1 - \frac{\alpha}{T_0} z\right) \left(\frac{\partial}{\partial t}\delta s + \frac{c_p}{T_0} X \delta v_z\right)$$
$$= -\alpha \nabla_l \delta\kappa_{lz} + \kappa^{st}_{lj} \nabla_l \nabla_j \delta T - \nabla_l \pi_l, \tag{38}$$

$$\frac{\partial}{\partial t}\delta n_i = \frac{1}{2}\lambda^{st}_{ijk} \partial_j \delta v_k + \frac{1}{\gamma_1}\left(\delta^{\perp}_{ik}\right)^{st} \delta h_k + \Upsilon_i. \tag{39}$$

A significant simplification of these equations is achived by noting that, on the one hand, for a typical thermotropic nematic, $\rho_0 \sim 1$, $T_0 \sim 10^2$, $\beta \sim 10^{-4}$, $c_T \sim 10^5$, $c_p \sim 10^7$ [29]. On the orher hand, in a typical light scattering experiment $\alpha \leq 1$, $z \leq 1$, with $g \sim 10^3$. As a consequence $\beta X z \lesssim 10^{-4}$, $\alpha z/T_0 \lesssim 10^{-2}$, $g\beta^2 T_0 z / [(\gamma - 1)\, c_p] = gz/c_T^2 \sim 10^{-7}$; accordingly, in Eqs. (36)-(39) the terms $\beta X z \lesssim 10^{-4}$, $gz/c_T^2 \sim 10^{-7}$ and $\alpha z/T_0 \lesssim 10^{-2}$ can be neglected. Thus, the set of equations (36)-(39) becomes the more compact set of nematodynamic fluctuating equations

$$\frac{\partial}{\partial t}\delta\rho = \rho_0 \beta \left[X + \frac{g\beta T_0}{(\gamma - 1)\, c_p}\right] \delta v_z - \rho_0 \nabla_l \delta v_l, \tag{40}$$

$$\rho_0 \frac{\partial}{\partial t}\delta v_i = -\nabla_i \delta p - \frac{1}{2}\lambda^{st}_{kji} \nabla_j \delta h_k + \nu^{st}_{ijlm} \nabla_j \nabla_m \delta v_l - g\delta_{iz}\delta\rho + \nabla_j \Sigma_{ij}, \tag{41}$$

$$\rho_0 T_0 \frac{\partial}{\partial t}\delta s = -\rho_0 c_p X \delta v_z - \alpha \nabla_l \delta\kappa_{lz} + \kappa^{st}_{lj} \nabla_l \nabla_j \delta T - \nabla_l \pi_l, \tag{42}$$

$$\frac{\partial}{\partial t}\delta n_i = \frac{1}{2}\lambda^{st}_{ijk} \nabla_j \delta v_k + \frac{1}{\gamma_1}\left(\delta^{\perp}_{ik}\right)^{st} \delta h_k + \Upsilon_i. \tag{43}$$

It is important emphasize several relevant features of these equations. First, if the nematic is incompressible, $\nabla_j \delta v_j = 0$, the density fluctuations do not vanish due to the presence of the external gradients. Secondly, they are consistent with known results in the literature, specifically, in the absence of a gravitational field ($g = 0$), if the isotropic limit of (40)-(43) is taken by setting $\delta n_i = 0$ and if the term $g\rho_0 \beta^2 T_0 \delta v_z / [(\gamma - 1)\, c_p] = \rho_0 g \delta v_z / c_T^2$ in Eq. (40) is eliminated, Eqs. (40)-(43) reduce to the corresponding hydrodynamic (non stochastic) equations for a simple fluid, Eqs. (2) in Ref. [25].

2.2 Pressure-entropy representation

Since the gravitational field induces a constant pressure gradient $\nabla_z p$ given by Eq. (2), and since in the geometry of the proposed model the director field initially has a preferential orientation $\widehat{n}_0$ along the z axis, the hydrodynamic variables may be divided into two independent sets which are transverse and longitudinal to $\widehat{n}_0$ and the wave vector $\overrightarrow{k}$, also defined in Figure 1. The former set is $\{v_x(\overrightarrow{r}, t), n_x(\overrightarrow{r}, t)\}$, while the latter one is $\{p(\overrightarrow{r},t), v_y(\overrightarrow{r},t), v_z(\overrightarrow{r},t), s(\overrightarrow{r},t), n_y(\overrightarrow{r},t)\}$. The corresponding linearized fluctuating hydrodynamic equations written in terms of these sets of variables, are easily obtained by using the thermodynamic relations

$$\delta\rho = \left(\frac{\partial\rho}{\partial p}\right)_s^{st}\delta p + \left(\frac{\partial\rho}{\partial s}\right)_p^{st}\delta s, \tag{44}$$

$$\delta T = \left(\frac{\partial T}{\partial p}\right)_s^{st}\delta p + \left(\frac{\partial T}{\partial s}\right)_p^{st}\delta s, \tag{45}$$

with $(\partial\rho/\partial p)_s^{st} \equiv 1/c_s^2$, $(\partial\rho/\partial s)_p^{st} \equiv -\beta^{st}\rho^{st}T^{st}/c_p = -\beta\rho_0 T_0/c_p$, $(\partial T/\partial p)_s^{st} = \beta T_0/(\rho_0 c_p)$, $(\partial T/\partial s)_p^{st} \equiv T^{st}/c_p = T_0/c_p$, being $\beta^{st} \equiv -1/\rho^{st}(\partial\rho/\partial T)_p^{st} = -1/\rho_0(\partial\rho/\partial T)_p$, $\chi_i^{st} \equiv \kappa_i/\rho^{st}c_p = \kappa_i/(\rho_0 c_p)$, for $i = \perp, \|$, $\kappa_a = \kappa_\| - \kappa_\perp$, the thermal diffusivity coefficient. In writing relations (44)-(45), it has been assumed that $\rho^{st} \simeq \rho_0$, $T^{st} \simeq T_0$ and that the thermodynamic quantities β, c_p, c_T, c_s, κ_i in the steady state, have the same values as in equilibrium. In this representation the complete set of linearized, fluctuating, hydrodynamic equations for $\{\delta p, \delta s, \delta v_i, \delta n_i\}$ is given by

$$\begin{aligned}\frac{\partial}{\partial t}\delta p &= \rho_0 g\delta v_z + (\gamma-1)\left[\chi_\perp\left(\nabla_x^2+\nabla_y^2\right)+\chi_\|\nabla_z^2\right]\delta p + \frac{\rho_0}{\beta}(\gamma-1)\left[\chi_\perp\left(\nabla_x^2+\nabla_y^2\right)\right.\\ &\left.+\chi_\|\nabla_z^2\right]\delta s - \rho_0 c_s^2\nabla_i\delta v_i - \alpha\beta\rho_0\chi_a c_s^2\left(\nabla_x\delta n_x+\nabla_y\delta n_y\right) - \frac{\gamma-1}{\beta T_0}\nabla_j\pi_j,\end{aligned} \tag{46}$$

$$\begin{aligned}\rho_0\frac{\partial}{\partial t}\delta v_x &= -\nabla_x\delta p + \left[(\nu_2+\nu_4)\nabla_x^2+\nu_2\nabla_y^2+\nu_3\nabla_z^2\right]\delta v_x + \nu_4\nabla_x\nabla_y\delta v_y\\ &+(\nu_3+\nu_5)\nabla_z\nabla_x\delta v_z - \frac{1}{2}(\lambda+1)(K_1\nabla_x^2+K_2\nabla_y^2+K_3\nabla_z^2)\nabla_z\delta n_x\\ &-\frac{1}{2}(\lambda+1)(K_1-K_2)\nabla_z\nabla_x\nabla_y\delta n_y + \nabla_j\Sigma_{xj},\end{aligned} \tag{47}$$

$$\begin{aligned}\rho_0\frac{\partial}{\partial t}\delta v_y &= -\nabla_y\delta p + \nu_4\nabla_y\nabla_x\delta v_x + \left[\nu_2\nabla_x^2+(\nu_2+\nu_4)\nabla_y^2+\nu_3\nabla_z^2\right]\delta v_y\\ &+(\nu_3+\nu_5)\nabla_z\nabla_y\delta v_z - \frac{1}{2}(\lambda+1)(K_1-K_2)\nabla_z\nabla_x\nabla_y\delta n_x\\ &-\frac{1}{2}(\lambda+1)\left(K_2\nabla_x^2+K_1\nabla_y^2+K_3\nabla_z^2\right)\nabla_z\delta n_y + \nabla_j\Sigma_{yj},\end{aligned} \tag{48}$$

$$\begin{aligned}\rho_0\frac{\partial}{\partial t}\delta v_z &= -\nabla_z\delta p - \frac{g}{c_s^2}\delta p + (\nu_3+\nu_5)\nabla_z\nabla_x\delta v_x + (\nu_3+\nu_5)\nabla_z\nabla_y\delta v_y + \left[\nu_3\left(\nabla_x^2+\nabla_y^2\right)\right.\\ &\left.+\left(2\nu_1+\nu_2-\nu_4+2\nu_5\right)\nabla_z^2\right]\delta v_z - \frac{1}{2}(\lambda-1)\left[K_1\left(\nabla_x^2+\nabla_y^2\right)+K_3\nabla_z^2\right]\nabla_x\delta n_x\\ &-\frac{1}{2}(\lambda-1)\left[K_1\left(\nabla_x^2+\nabla_y^2\right)+K_3\nabla_z^2\right]\nabla_y\delta n_y + g\frac{\beta\rho_0 T_0}{c_p}\delta s + \nabla_j\Sigma_{zj},\end{aligned} \tag{49}$$

$$\begin{aligned}\frac{\partial}{\partial t}\delta s &= -\frac{c_p}{T_0}X\delta v_z + \frac{\beta}{\rho_0}\left[\chi_\perp\left(\nabla_x^2+\nabla_y^2\right)+\chi_\|\nabla_z^2\right]\delta p + \left[\chi_\perp\left(\nabla_x^2+\nabla_y^2\right)\right.\\ &\left.+\chi_\|\nabla_z^2\right]\delta s - \alpha\frac{\chi_a c_p}{T_0}\left(\nabla_x\delta n_x+\nabla_y\delta n_y\right) - \frac{1}{\rho_0 T_0}\nabla_j\pi_j,\end{aligned} \tag{50}$$

$$\frac{\partial}{\partial t}\delta n_x = \frac{1}{2}\left(\lambda - 1\right)\nabla_x \delta v_z + \frac{1}{2}\left(\lambda + 1\right)\nabla_z \delta v_x + \frac{1}{\gamma_1}(K_1 - K_2)\nabla_x \nabla_y \delta n_y$$
$$+ \frac{1}{\gamma_1}(K_1 \nabla_x^2 + K_2 \nabla_y^2 + K_3 \nabla_z^2)\delta n_x + \Upsilon_x, \tag{51}$$

$$\frac{\partial}{\partial t}\delta n_y = \frac{1}{2}\left(\lambda + 1\right)\nabla_z \delta v_y + \frac{1}{2}\left(\lambda - 1\right)\nabla_y \delta v_z + \frac{1}{\gamma_1}(K_1 - K_2)\nabla_x \nabla_y \delta n_x$$
$$+ \frac{1}{\gamma_1}(K_2 \nabla_x^2 + K_1 \nabla_y^2 + K_3 \nabla_z^2)\delta n_y + \Upsilon_y, \tag{52}$$

whith $j = x,\ y,\ z$, and where the FDR of the stochastic components of the fluxes are given by Eqs. (33)-(35).

2.3 Symmetry breaking representation

For the purpose of calculating the spectrum of light scattering of the nematic, it will be convenient to introduce a different set of fluctuating thermodynamic variables that takes into account the effect of the intrinsec anisotropy of the fluid. A proper set of new state variables that describe the dynamics of fluctuations in a simple fluid with broken symmetry along the z axis, was proposed long ago in Ref. [25]. Actually, this is the case of the nematic layer under consideration, because owing to the initial orientation of the director $\widehat{n}_i^{st}$, the NLC exhibits several symmetries, namely, rotational invariances around the z axis, under inversions with respect to the xy plane and at reflections on planes containing the z axis. Thus, following Ref. [25] we introduce the set of variables $\{\delta\varphi, \delta\psi, \delta\xi, \delta f_1, \delta f_2, \delta s, \delta p\}$ defined as follows [34],

$$\delta\varphi \equiv \nabla \cdot \delta\overrightarrow{v} = \nabla_x \delta v_x + \nabla_y \delta v_y + \nabla_z \delta v_z, \tag{53}$$

$$\delta\psi \equiv (\nabla \times \delta\overrightarrow{v})_z = \nabla_x \delta v_y - \nabla_y \delta v_x, \tag{54}$$

$$\delta\xi \equiv (\nabla \times \nabla \times \delta\overrightarrow{v})_z = \frac{\partial \delta\varphi}{\partial z} - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\delta v_z. \tag{55}$$

By analogy, the director deviations $\delta\overrightarrow{n}$ are

$$\delta f_1 \equiv \nabla \cdot \delta\overrightarrow{n} = \nabla_x \delta n_x + \nabla_y \delta n_y, \tag{56}$$

$$\delta f_2 \equiv (\nabla \times \delta\overrightarrow{n})_z = \nabla_x \delta n_y - \nabla_y \delta n_x. \tag{57}$$

In this new representation the complete set of hydrodynamic equations (46)-(52) takes the form

$$\frac{\partial}{\partial t}\delta p = (\gamma - 1)(\chi_\perp \nabla_\perp^2 + \chi_\parallel \nabla_\parallel^2)\delta p + \frac{\rho_0}{\beta}(\gamma - 1)(\chi_\perp \nabla_\perp^2 + \chi_\parallel \nabla_\parallel^2)\delta s$$
$$+ g\rho_0 \delta v_z - \rho_0 c_s^2 \delta\varphi - \alpha\beta\rho_0\chi_a c_s^2 \delta f_1 - \frac{\gamma - 1}{\beta T_0}\nabla_j \pi_j, \tag{58}$$

$$\rho_0 \frac{\partial}{\partial t}\delta\varphi = [(2\nu_3 - \nu_2 - \nu_4 + \nu_5)\nabla_\perp^2 + (2\nu_1 + \nu_2 - 2\nu_3 - \nu_4 + \nu_5)\nabla_\parallel^2]\nabla_z \delta v_z$$
$$- \left(\nabla^2 + \frac{g}{c_s^2}\nabla_z\right)\delta p + g\frac{\beta\rho_0 T_0}{c_p}\nabla_z \delta s + \left[(\nu_2 + \nu_4)\nabla_\perp^2 + (2\nu_3 + \nu_5)\nabla_\parallel^2\right]\delta\varphi$$
$$-\lambda(K_1\nabla_\perp^2 + K_3\nabla_\parallel^2)\nabla_z \delta f_1 + \nabla_j\left(\nabla_x \Sigma_{xj} + \nabla_y \Sigma_{yj} + \nabla_z \Sigma_{zj}\right), \tag{59}$$

$$\frac{\partial}{\partial t}\delta s = \frac{\beta}{\rho_0}\left(\chi_\perp \nabla_\perp^2 + \chi_\parallel \nabla_\parallel^2\right)\delta p + \left(\chi_\perp \nabla_\perp^2 + \chi_\parallel \nabla_\parallel^2\right)\delta s$$
$$-\frac{c_p}{T_0} X \delta v_z - \alpha\frac{\chi_a c_p}{T_0}\delta f_1 - \frac{1}{\rho_0 T_0}\nabla_j \pi_j, \tag{60}$$

$$\rho_0 \frac{\partial}{\partial t}\delta\xi = \frac{g}{c_s^2}\nabla_\perp^2 \delta p - g\frac{\beta\rho_0 T_0}{c_p}\nabla_\perp^2 \delta s + \left[(\nu_2 - \nu_3 + \nu_4 - \nu_5)\nabla_\perp^2 + \nu_3 \nabla_\parallel^2\right]\nabla_z \delta\varphi$$
$$+\frac{1}{2}[(\lambda - 1)\nabla_\perp^2 - (\lambda + 1)\nabla_\parallel^2](K_1\nabla_\perp^2 + K_3\nabla_\parallel^2)\delta f_1 - [\nu_3(\nabla_\perp^2 - \nabla_\parallel^2)^2$$
$$+2(\nu_1 + \nu_2)\nabla_\perp^2\nabla_\parallel^2]\delta v_z + \nabla_z \nabla_j\left(\nabla_x \Sigma_{xj} + \nabla_y \Sigma_{yj} + \nabla_z \Sigma_{zj}\right) - \nabla^2 \nabla_j \Sigma_{zj}, \tag{61}$$

$$\frac{\partial}{\partial t}\delta f_1 = \frac{1}{2}\left[(\lambda - 1)\nabla_\perp^2 - (\lambda + 1)\nabla_z^2\right]\delta v_z + \frac{1}{\gamma_1}(K_1\nabla_\perp^2 + K_3\nabla_\parallel^2)\delta f_1$$
$$+\frac{1}{2}(\lambda + 1)\nabla_z \delta\varphi + \nabla_x \Upsilon_x + \nabla_y \Upsilon_y, \tag{62}$$

$$\rho_0 \frac{\partial}{\partial t}\delta\psi = (\nu_2 \nabla_\perp^2 + \nu_3 \nabla_\parallel^2)\delta\psi - \frac{1}{2}(\lambda + 1)(K_2\nabla_\perp^2 + K_3\nabla_\parallel^2)\nabla_z \delta f_2$$
$$+\nabla_j\left(\nabla_x \Sigma_{yj} - \nabla_y \Sigma_{xj}\right), \tag{63}$$

$$\frac{\partial}{\partial t}\delta f_2 = \frac{1}{2}(\lambda + 1)\nabla_z \delta\psi + \frac{1}{\gamma_1}(K_2\nabla_\perp^2 + K_3\nabla_\parallel^2)\delta f_2$$
$$+\nabla_x \Upsilon_y - \nabla_y \Upsilon_x, \tag{64}$$

where $\nabla_\perp^2 \equiv \nabla_x^2 + \nabla_y^2$, $\nabla_\parallel^2 \equiv \nabla_z^2$. In Eqs. (58)-(64) δv_z is coupled to $\delta\xi$ and $\delta\varphi$ through the relation

$$\delta\xi \equiv \nabla_z \delta\varphi - \nabla^2 \delta v_z. \tag{65}$$

Furthermore, if the Fourier transform of an arbitrary field $A(\overrightarrow{r}, t)$ with respect to $\overrightarrow{r}$ is defined by

$$\widetilde{A}(\overrightarrow{k}, \omega) \equiv \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} A(\overrightarrow{r}, t)\exp\left[-i(\overrightarrow{k}\cdot\overrightarrow{r} - \omega t)\right] d\overrightarrow{r} dt, \tag{66}$$

with

$$A(\overrightarrow{r}, t) = \int_{-\infty}^{\infty} \widetilde{A}(\overrightarrow{k}, \omega)\exp\left[i(\overrightarrow{k}\cdot\overrightarrow{r} - \omega t)\right] d\overrightarrow{k} d\omega, \tag{67}$$

in matrix form the transformed set of Eqs. (58)-(64) reads

$$\frac{\partial}{\partial t}\delta\overrightarrow{X}(\overrightarrow{k},t) = -M\delta\overrightarrow{X}(\overrightarrow{k},t) + \overrightarrow{\Theta}(\overrightarrow{k},t), \tag{68}$$

where

$$\delta\overrightarrow{X}(\overrightarrow{k},t) = \left(\delta\overrightarrow{X}^{L}, \delta\overrightarrow{X}^{T}\right)^{t} \tag{69}$$

and

$$\delta\overrightarrow{X}^{L}(\overrightarrow{k},t) = \left(\delta\widetilde{p}, \delta\widetilde{\varphi}, \delta\widetilde{s}, \delta\widetilde{\xi}, \delta\widetilde{f}_1\right)^{t}, \tag{70}$$

$$\delta\overrightarrow{X}^{T}(\overrightarrow{k},t) = \left(\delta\widetilde{\psi}, \delta\widetilde{f}_2\right)^{t}. \tag{71}$$

The hydrodynamic matrix M is diagonal by blocks,

$$M = \left(\begin{array}{c|c} M^{L} & 0 \\ \hline 0 & M^{T} \end{array}\right). \tag{72}$$

where the superscripts L and T denote, respectively, the longitudinal and transverse sets of variables. The explicit form of the submatrices M^L, M^T, is

$$M^{L} = \begin{pmatrix} (\gamma-1)\,D_T k^2 & \rho_0 c_s^2 + g\frac{\rho_0 i k_z}{k^2} & \frac{\rho_0}{\beta}(\gamma-1)\,D_T k^2 & -g\frac{\rho_0}{k^2} & \alpha\beta\rho_0\chi_a c_s^2 \\ -\frac{k^2}{\rho_0} + g\frac{ik_z}{\rho_0 c_s^2} & \sigma_1 k^2 & -g\frac{\beta T_0}{c_p} i k_z & \sigma_2 i k_z & -\frac{\lambda K_I}{\rho_0} i k^2 k_z \\ \frac{\beta}{\rho_0} D_T k^2 & -X\frac{c_p i k_z}{T_0 k^2} & D_T k^2 & X\frac{c_p}{T_0 k^2} & \alpha\frac{\chi_a c_p}{T_0} \\ g\frac{k_\perp^2}{\rho_0 c_s^2} & -\sigma_2 i k_\perp^2 k_z & -g\frac{\beta T_0}{c_p} k_\perp^2 & \sigma_3 k^2 & -\frac{\Omega}{\rho_0} K_I k^4 \\ 0 & -\lambda\frac{i k_\perp^2 k_z}{k^2} & 0 & \Omega & \frac{K_I}{\gamma_1} k^2 \end{pmatrix} \tag{73}$$

and

$$M^{T} = \begin{pmatrix} \sigma_4 k^2 & -\frac{\lambda_+ K_{II}}{\rho_0} i k^2 k_z \\ -\lambda_+ i k_z & \frac{K_{II}}{\gamma_1} k^2 \end{pmatrix}.$$

with

$$D_T \equiv \frac{1}{k^2}\left(\chi_\perp k_\perp^2 + \chi_\parallel k_\parallel^2\right), \tag{74}$$

$$\sigma_1 \equiv \frac{1}{\rho_0 k^4}[(\nu_2+\nu_4)\,k_\perp^4 + 2\,(2\nu_3+\nu_5)\,k_\parallel^2 k_\perp^2 + (2\nu_1+\nu_2-\nu_4+2\nu_5)\,k_\parallel^4], \tag{75}$$

$$\sigma_2 \equiv \frac{1}{\rho_0 k^2}\left[(-\nu_2+2\nu_3-\nu_4+\nu_5)\,k_\perp^2 + (2\nu_1+\nu_2-2\nu_3-\nu_4+\nu_5)\,k_\parallel^2\right], \tag{76}$$

$$\sigma_3 \equiv \frac{1}{\rho_0 k^4}\left[2\,(\nu_1+\nu_2)\,k_\parallel^2 k_\perp^2 + \nu_3\left(k_\parallel^2 - k_\perp^2\right)^2\right], \tag{77}$$

$$\sigma_4 \equiv \frac{1}{\rho_0 k^2}\left(\nu_2 k_\perp^2 + \nu_3 k_\parallel^2\right), \tag{78}$$

$$K_I \equiv \frac{1}{k^2}\left(K_1 k_\perp^2 + K_3 k_\parallel^2\right), \tag{79}$$

$$K_{II} \equiv \frac{1}{k^2}\left(K_2 k_\perp^2 + K_3 k_\parallel^2\right), \tag{80}$$

$$\Omega \equiv \frac{1}{k^2}\left(\lambda_- k_\perp^2 - \lambda_+ k_\parallel^2\right), \tag{81}$$

$$\lambda_- \equiv \frac{1}{2}(\lambda - 1), \qquad \lambda_+ \equiv \frac{1}{2}(\lambda + 1). \tag{82}$$

It should be noted that D_T has the dimensions and values of the orders of magnitude of the coefficients of thermal diffusivity $\chi_\perp, \chi_\parallel$. The quantities σ_1, σ_2, σ_3, σ_4, have values comparable to the coefficients ν_j/ρ_0 (for $j = 1\ldots5$); while K_I, K_{II}, have similar values to those of the elastic constants K_j (for $j = 1, 2, 3$). Finally, the dimensionless quantity Ω is a function of the previously defined dimensionless coefficient λ and is a measure of the anisotropy of the nematic.

The statistical terms in Eq. (68) are given by the column vector

$$\vec{\Theta}(\vec{k}, t) = \left(\vec{\Theta}^L, \vec{\Theta}^T\right)^t, \tag{83}$$

where the superscript t denotes the transpose. Also

$$\vec{\Theta}^L(\vec{k}, t) = \begin{pmatrix} -\frac{i(\gamma-1)}{\beta T_0} k_j \widetilde{\pi}_j \\ -\frac{k_j}{\rho_0}\left(k_x \widetilde{\Sigma}_{xj} + k_y \widetilde{\Sigma}_{yj} + k_z \widetilde{\Sigma}_{zj}\right) \\ -\frac{ik_l}{\rho_0 T_0}\widetilde{\pi}_l \\ -\frac{ik_z k_j}{\rho_0}\left(k_x \widetilde{\Sigma}_{xj} + k_y \widetilde{\Sigma}_{yj} + k_z \widetilde{\Sigma}_{zj}\right) + \frac{ik^2}{\rho_0} k_j \widetilde{\Sigma}_{zj} \\ ik_x \Upsilon_x + ik_y \Upsilon_y \end{pmatrix}, \tag{84}$$

$$\vec{\Theta}^T(\vec{k}, t) = \begin{pmatrix} -\frac{k_j}{\rho_0}\left(k_x \widetilde{\Sigma}_{yj} - k_y \widetilde{\Sigma}_{xj}\right) \\ ik_x \widetilde{\Upsilon}_y - ik_y \widetilde{\Upsilon}_x \end{pmatrix}. \tag{85}$$

As a result of this change of representation, the original system of Eqs. (58)-(64) is simplified into two uncoupled systems of equations, namely, five equations for the longitudinal variables $\delta\vec{X}^L$, Eq. (70), and two equations for the transverse variables $\delta\vec{X}^T$, Eq. (71).

2.3.1 Equilibrium

If the nonequilibrium terms containing α and g are neglected in Eqs. (68), the resulting equations describe the equilibrium state. In this case the hydrodynamic matrix is given by

$$M_E = \left(\begin{array}{c|c} M_L^E & 0 \\ \hline 0 & M_T^E \end{array}\right), \tag{86}$$

with

$$M_E^L = \begin{pmatrix} (\gamma-1)D_T k^2 & \rho_0 c_s^2 & \frac{\rho_0}{\beta}(\gamma-1)D_T k^2 & 0 & 0 \\ -\frac{k^2}{\rho_0} & \sigma_1 k^2 & 0 & \sigma_2 i k_z & -\frac{\lambda K_I}{\rho_0} i k^2 k_z \\ \frac{\beta}{\rho_0} D_T k^2 & 0 & D_T k^2 & 0 & 0 \\ 0 & -\sigma_2 i k_\perp^2 k_z & 0 & \sigma_3 k^2 & -\frac{\Omega}{\rho_0} K_I k^4 \\ 0 & -\lambda \frac{i k_\perp^2 k_z}{k^2} & 0 & \Omega & \frac{K_I}{\gamma_1} k^2 \end{pmatrix} \tag{87}$$

and

$$M_E^T = \begin{pmatrix} \sigma_4 k^2 & -\frac{\lambda_+ K_{II}}{\rho} i k^2 k_z \\ -\lambda_+ i k_z & \frac{K_{II}}{\gamma_1} k^2 \end{pmatrix}. \tag{88}$$

Note that in Eq. (86) still prevails the same structure by blocks shown in Eq. (72), that is, in the equilibrium state longitudinal and transverse variables are completely decoupled; furthermore, M_E^L contains more null entries and is simpler than M^L. On the other hand, M_E^T is identical to M^T. Thus, the nonequilibrium effects caused by the presence of α and g, only affect the longitudinal variables.

3 Hydrodynamic modes

In order to facilitate the calculation of hydrodynamic modes, we define the following variables of the same dimension, $[\delta z_j] = M^{1/2} L^{-1/2} t$ (for $j = 1, \ldots, 7$),

$$\begin{aligned} z_1(\overrightarrow{k}, t) &\equiv \left(\frac{1}{\rho_0 c_s^2}\right)^{1/2} \delta\widetilde{p}, & z_5(\overrightarrow{k}, t) &\equiv \left(\frac{\rho_0 c_s^2}{k^2}\right)^{1/2} \delta\widetilde{f}_1, \\ z_2(\overrightarrow{k}, t) &\equiv \left(\frac{\rho_0}{k^2}\right)^{1/2} \delta\widetilde{\varphi}, & z_6(\overrightarrow{k}, t) &\equiv \left(\frac{\rho_0}{k^2}\right)^{1/2} \delta\widetilde{\psi}, \\ z_3(\overrightarrow{k}, t) &\equiv \left(\frac{\rho_0 T_0}{c_p}\right)^{1/2} \delta\widetilde{s}, & z_7(\overrightarrow{k}, t) &\equiv \left(\frac{\rho_0 c_s^2}{k^2}\right)^{1/2} \delta\widetilde{f}_2. \\ z_4(\overrightarrow{k}, t) &= \left(\frac{\rho_0}{k^4}\right)^{1/2} \delta\widetilde{\xi}, \end{aligned} \tag{89}$$

The system (68) expressed in terms of the variables (89) is rewritten as

$$\frac{\partial}{\partial t} \overrightarrow{Z}(\overrightarrow{k}, t) = -N \overrightarrow{Z}(\overrightarrow{k}, t) + \overrightarrow{\Xi}(\overrightarrow{k}, t), \tag{90}$$

in which

$$\overrightarrow{Z}(\overrightarrow{k}, t) = \left(\overrightarrow{Z}^L, \overrightarrow{Z}^T\right)^t \tag{91}$$

is the vector of variables of the same size, formed by the longitudinal

$$\overrightarrow{Z}^L(\overrightarrow{k}, t) = (z_1, z_2, z_3, z_4, z_5)^t \tag{92}$$

and transverse

$$\overrightarrow{Z}^T(\overrightarrow{k}, t) = (z_6, z_7)^t \tag{93}$$

variables. The hydrodynamic matrix N

$$N = \left(\begin{array}{c|c} N^L & 0 \\ \hline 0 & N^T \end{array}\right) \tag{94}$$

is composed by the submatrices

$$N^L = \begin{pmatrix} (\gamma-1) D_T k^2 & c_s k + \frac{g}{c_s}\frac{ik_z}{k} & (\gamma-1)^{1/2} D_T k^2 & -\frac{g}{c_s} & \alpha\beta\chi_a k \\ -c_s k + \frac{g}{c_s}\frac{ik_z}{k} & \sigma_1 k^2 & -(\gamma-1)^{1/2} \frac{g}{c_s}\frac{ik_z}{k} & \sigma_2 ikk_z & -\frac{\lambda K_I}{\rho_0 c_s} ik^2 k_z \\ (\gamma-1)^{1/2} D_T k^2 & -\frac{\beta X c_s}{(\gamma-1)^{1/2}}\frac{ik_z}{k} & D_T k^2 & \frac{\beta X c_s}{(\gamma-1)^{1/2}} & \frac{\alpha\beta\chi_a}{(\gamma-1)^{1/2}} k \\ \frac{g}{c_s}\frac{k_\perp^2}{k^2} & -\sigma_2 \frac{ik_\perp^2 k_z}{k} & -(\gamma-1)^{1/2}\frac{g}{c_s}\frac{k_\perp^2}{k^2} & \sigma_3 k^2 & -\frac{\Omega K_I}{\rho_0 c_s} k^3 \\ 0 & -\lambda c_s \frac{ik_\perp^2 k_z}{k^2} & 0 & \Omega c_s k & \frac{K_I}{\gamma_1} k^2 \end{pmatrix} \tag{95}$$

and

$$N^T = \begin{pmatrix} \sigma_4 k^2 & -\frac{\lambda_+ K_{II}}{\rho_0 c_s} ik^2 k_z \\ -\lambda_+ c_s ik_z & \frac{K_{II}}{\gamma_1} k^2 \end{pmatrix}. \tag{96}$$

It can also easily verified that the dimension of each input N_{ij} of the matrix N is $[N_{ij}] = t^{-1}$.

Moreover, in Eq. (90) the stochastic vectors

$$\overrightarrow{\Xi}(\overrightarrow{k}, t) = \left(\overrightarrow{\Xi}^L, \overrightarrow{\Xi}^T\right)^t, \tag{97}$$

are composed of the longitudinal

$$\overrightarrow{\Xi}^L(\overrightarrow{k}, t) = (\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5)^t \tag{98}$$

and the transverse

$$\overrightarrow{\Xi}^T(\overrightarrow{k}, t) = (\zeta_6, \zeta_7)^t \tag{99}$$

noise vectors. The stochastic noise components ζ_m, $m = 1 \ldots 6$, indicated in each one of them, are given explicitly in Appendix A. By taking into account the fluctuation-dissipation relations Eqs. (33)-(34), the autocorrelations and cross-correlations functions of the stochastic noises, Eqs. (169)-(175), averaged over the steady state are also given in Appendix A.

In order to find hydrodynamic modes of the linear system (90), it is required to calculate its eigenvalues (λ), which are given by the roots of the characteristic equation

$$p(\lambda) = p^L(\lambda) p^T(\lambda) = 0, \tag{100}$$

where $p^L(\lambda)$ and $p^T(\lambda)$ are the characteristic polynomials of fifth and second order in λ of the matrices N^L and N^T, respectively. These roots are calculated below.

3.1 Longitudinal modes

To simplify the calculation of $p_L(\lambda)$ of the matrix N^L we first define

$$\overrightarrow{Z}^L = \left(\overrightarrow{Z}^L_X, \overrightarrow{Z}^L_Y\right)^t \tag{101}$$

with

$$\overrightarrow{Z}^L_X = (z_1, z_2)^t \tag{102}$$

and

$$\overrightarrow{Z}^L_Y = (z_3, z_4, z_5)^t. \tag{103}$$

Then, from Eq. (90) we obtain the system

$$\frac{\partial}{\partial t} \overrightarrow{Z}^L(\overrightarrow{k}, t) = -N^L \overrightarrow{Z}^L(\overrightarrow{k}, t) + \overrightarrow{\Xi}^L(\overrightarrow{k}, t), \tag{104}$$

where the matrix of coefficients is

$$N^L = \left(\begin{array}{c|c} N^L_{XX} & N^L_{XY} \\ \hline N^L_{YX} & N^L_{YY} \end{array} \right). \tag{105}$$

The submatrices are defined as

$$N^L_{XX} = \begin{pmatrix} (\gamma-1)\, D_T k^2 & c_s k + \frac{g}{c_s}\frac{ik_z}{k} \\ -c_s k + \frac{g}{c_s}\frac{ik_z}{k} & \sigma_1 k^2 \end{pmatrix}, \tag{106}$$

$$N^L_{XY} = \begin{pmatrix} (\gamma-1)^{1/2} D_T k^2 & -\frac{g}{c_s} & \alpha\beta\chi_a k \\ -(\gamma-1)^{1/2}\frac{g}{c_s}\frac{ik_z}{k} & \sigma_2 ikk_z & -\frac{\lambda K_I}{\rho_0 c_s} ik^2 k_z \end{pmatrix}, \tag{107}$$

$$N^L_{YX} = \begin{pmatrix} (\gamma-1)^{1/2} D_T k^2 & -\frac{\beta X c_s}{(\gamma-1)^{1/2}}\frac{ik_z}{k} \\ \frac{g}{c_s}\frac{k_\perp^2}{k^2} & -\sigma_2 \frac{ik_\perp^2 k_z}{k} \\ 0 & -\lambda c_s \frac{ik_\perp^2 k_z}{k^2} \end{pmatrix} \tag{108}$$

and

$$N^L_{YY} = \begin{pmatrix} D_T k^2 & \frac{\beta X c_s}{(\gamma-1)^{1/2}} & \frac{\alpha\beta\chi_a}{(\gamma-1)^{1/2}} k \\ -(\gamma-1)^{1/2}\frac{g}{c_s}\frac{k_\perp^2}{k^2} & \sigma_3 k^2 & -\frac{\Omega K_I}{\rho_0 c_s} k^3 \\ 0 & \Omega c_s k & \frac{K_I}{\gamma_1} k^2 \end{pmatrix}. \tag{109}$$

Furthermore,

$$\overrightarrow{\Xi}^L(\overrightarrow{k}, t) = \left(\overrightarrow{\Xi}^L_X, \overrightarrow{\Xi}^L_Y\right)^t \tag{110}$$

is the vector of longitudinal stochastic terms, with components

$$\overrightarrow{\Xi}^L_X(\overrightarrow{k}, t) = (\zeta_1, \zeta_2)^t \tag{111}$$

and

$$\overrightarrow{\Xi}^L_Y(\overrightarrow{k}, t) = (\zeta_3, \zeta_4, \zeta_5)^t . \tag{112}$$

Following the method proposed by [35] for a simple fluid, it can be shown the system Eq. (104) has the property that, within a very good approximation, the variables $\delta\overrightarrow{Z}^L_X$ and $\delta\overrightarrow{Z}^L_Y$ are mutually independent [21]. This statement implies that in the matrix N^L the blocks N^L_{XY} and N^L_{YX} can be neglected and Eq. (105) is simplified to

$$N^L = \left(\begin{array}{c|c} N^L_{XX} & 0 \\ \hline 0 & N^L_{YY} \end{array} \right). \tag{113}$$

Consequently, the set of equations (104) is reduced to the uncoupled system

$$\frac{\partial}{\partial t}\overrightarrow{Z}^L_X(\overrightarrow{k}, t) = -N^L_{XX}\overrightarrow{Z}^L_X(\overrightarrow{k}, t) + \overrightarrow{\Xi}^L_X(\overrightarrow{k}, t), \tag{114}$$

$$\frac{\partial}{\partial t}\overrightarrow{Z}^L_Y(\overrightarrow{k}, t) = -N^L_{YY}\overrightarrow{Z}^L_Y(\overrightarrow{k}, t) + \overrightarrow{\Xi}^L_Y(\overrightarrow{k}, t). \tag{115}$$

The same approximatiom allows to rewrite the characteristic polynomial of longitudinal variables as

$$p^L(\lambda) = p^L_{XX}(\lambda) p^L_{YY}(\lambda), \tag{116}$$

where

$$\begin{aligned} p^L_{XX}(\lambda) &= \lambda^2 - \left[(\gamma - 1)\, D_T k^2 + \sigma_1 k^2\right] \lambda \\ &+ (\gamma - 1)\, \sigma_1 k^2 D_T k^2 + k^2 c_s^2 + \frac{g^2}{c_s^2} \frac{k_z^2}{k^2} \end{aligned} \tag{117}$$

and

$$\begin{aligned} p^L_{YY}(\lambda) = \lambda^3 &- \left(D_T k^2 + \sigma_3 k^2 + \frac{K_I k^2}{\gamma_1} \right) \lambda^2 + \left(D_T k^2 \sigma_3 k^2 + D_T k^2 \frac{K_I k^2}{\gamma_1} \right. \\ &\left. + \sigma_3 k^2 \frac{K_I k^2}{\gamma_1} + \frac{\Omega^2 K_I k^4}{\rho_0} + g X \beta \frac{k_\perp^2}{k^2} \right) \lambda - D_T k^2 \sigma_3 k^2 \frac{K_I k^2}{\gamma_1} \\ &- D_T k^2 \frac{\Omega^2 K_I k^4}{\rho_0} - g X \beta \frac{k_\perp^2}{k^2} \frac{K_I k^2}{\gamma_1} + g \alpha \beta \frac{k_\perp^2}{k^2} \Omega \chi_a k^2. \end{aligned} \tag{118}$$

While there is no analytical difficulty to solve the quadratic and cubic equations (117) and (118), the explicit form of their exact roots can be quite complicated, especially for the latter. However, it is possible to estimate them following a procedure based partially on a method suggested in Ref. [36]. According to it, in Eq. (117) the following quantities $(\gamma - 1)\, D_T k^2$, $\sigma_1 k^2$, $k^2 c_s^2$ y $g^2 k_z^2/(c_s^2 k^2)$, may be identified. They depend on the thermal diffusion coefficient D_T, the viscosity σ_1, as well as on the gravitational field g and the adiabatic speed of sound propagation c_s. On the other hand, in Eq. (118) the quantities $g\alpha\beta \frac{k_\perp^2}{k^2}$, $gX\beta \frac{k_\perp^2}{k^2}$, $D_T k^2$, $\Omega \chi_a k^2$, $\sigma_3 k^2$, $\frac{K_1}{\gamma_1} k^2$ and $\frac{\Omega^2 K_I}{\rho_0} k^4$, may be also identified. They depend on both the, nematic material parameters, as the coefficients of thermal diffusivity $\chi_\parallel$, $\chi_\perp$, the viscosity coefficient ν_3, the elastic constants K_1,K_3, as well as on the temperature gradient α and the gravitational field g. It is helpful to compare these quantities with $\omega \equiv c_s k$, by introducing the small or reducted quantities

$$\begin{aligned} a_0 \equiv \frac{g\alpha\beta}{\omega} \frac{k_\perp^2}{k^2}, \quad a_0' \equiv \frac{gX\beta}{\omega} \frac{k_\perp^2}{k^2}, \quad a_0'' \equiv \frac{g^2 k_z^2}{\omega c_s^2 k^2}, \quad a_1 \equiv \frac{D_T k^2}{\omega}, \quad a_1' \equiv \frac{\Omega \chi_a k^2}{\omega}, \\ a_2 \equiv \frac{\sigma_1 k^2}{\omega}, \quad a_3 \equiv \frac{\sigma_3 k^2}{\omega}, \quad a_5 \equiv \frac{K_I k^2}{\gamma_1 \omega}, \quad a_6 \equiv \frac{\Omega^2 K_I k^4}{\rho_0 \omega}. \end{aligned} \tag{119}$$

For most nematics at ambient temperatures, ρ_0 and Ω are of order of magnitude 1, $\gamma_1 \sim 10^{-1}$, χ_i and ν_i are of order $10^{-2} - 10^{-3}$, $K_i \sim 10^{-6} - 10^{-7}$, while $\beta \sim 10^{-4}$ [29]; also, we consider that $\alpha \lesssim 1$ and $g \sim 10^3$. Since in typical light scattering experiments $k = 10^5 cm^{-1}$ and $c_s = 1.5 \times 10^5 cm s^{-1}$ [37], [38], the quantities given in Eq. (119) have the following orders of magnitude: $a_0 \sim 10^{-11}$, $a_0' \sim 10^{-11}$, $a_0'' \sim 10^{-14}$, $a_1 \sim 10^{-3}$, $a_1' \sim 10^{-3}$, $a_2 \sim 10^{-2}$, $a_3 \sim 10^{-2}$, $a_5 \sim 10^{-5}$ and $a_6 \sim 10^4$. If we were to follow the method of Ref. [36], the solutions of Eqs. (117) and (118) should be obtained by a perturbative approximation in terms of these small quantities. However, we will improve this approximation by using the exact roots of Eqs. (117) and (118) and by expressing them in terms of reduced quantities (119) of lower order in k^2 [21]. This procedure will be implemented in the next two subsections.

3.1.1 Sound longitudinal modes

In accordance with Eq. (117), the sound propagation modes are the roots of the characteristic equation $p^L_{XX}(\lambda) = 0$. In terms of the variable $s \equiv \lambda/\omega$ and the small quantities given in Eq. (119), this characteristic equation is rewritten as

$$s^2 + A's + B' = 0, \tag{120}$$

where

$$A' \equiv -\left[(\gamma - 1)\, a_1 + a_2\right], \tag{121}$$

$$B' \equiv 1 + (\gamma - 1)\, a_1 a_2 + \frac{a_0''}{\omega}. \tag{122}$$

Analytical solutions of Eq. (120) are

$$s_+ \simeq -\frac{1}{2}A' + \frac{1}{2}\sqrt{\Delta'}, \tag{123}$$

$$s_- \simeq -\frac{1}{2}A' - \frac{1}{2}\sqrt{\Delta'}, \tag{124}$$

in which

$$\Delta' \equiv A'^2 - 4B' \tag{125}$$

is the discriminant. Its sign determines the nature of the roots (123) and (124), which can only present one of the following three characteristics: two real and distinct roots, if $\Delta' > 0$; two real and equal roots, if $\Delta' = 0$ and two complex conjugate roots, when $\Delta' < 0$. Thus, according to the orders of magnitude of small amounts (119) in the coefficients (121) and (122), the discriminant (125) can be simplified to $\Delta' \simeq -4k^2c_s^2$, given that a_2, a_1, $a_0''/\omega^2 \ll 1$. In fact, $\Delta' < 0$ always. Note that since $a_0''/\omega \sim 10^{-24}$, the effect of external gravitational field g in Δ' is negligible. Therefore, solutions (123) and (124) will be complex conjugate,

$$s_+ \simeq \frac{1}{2}\left[(\gamma - 1)\, a_1 + a_2\right] + i, \tag{126}$$

$$s_- \simeq \frac{1}{2}\left[(\gamma - 1)\, a_1 + a_2\right] - i. \tag{127}$$

Rewriting these roots in terms of the variables λ_i by means of the relation $\lambda \equiv \omega s$, leads to

$$\lambda_1 \simeq \Gamma k^2 + ic_s k, \tag{128}$$

$$\lambda_2 \simeq \Gamma k^2 - ic_s k, \tag{129}$$

where

$$\Gamma \equiv \frac{1}{2}\left[(\gamma-1)\,D_T+\sigma_1\right] \tag{130}$$

is the sound attenuation coefficient of the nematic fluid. It should be noted that the sound propagation modes found, Eqs. (128) and (129), are in complete agreement with those already reported in the literature for NLC [27], [39].

3.1.2 Thermal diffusive, shear and director longitudinal modes

According to Eq. (116), the thermal diffusive, shear and director modes, are the roots of the characteristic equation $p^L_{YY}(\lambda)=0$. Again, in terms of the variable $s\equiv\lambda/\omega$ and the small quantities (119), this equation reads

$$s^3+As^2+Bs+C=0, \tag{131}$$

where

$$A\equiv -a_1-a_3-a_5, \tag{132}$$

$$B\equiv a_1a_3+a_1a_5+a_3a_5+\frac{a_6}{\omega}+\frac{a_0'}{\omega}, \tag{133}$$

$$C\equiv -a_1a_3a_5-\frac{a_1a_6}{\omega}-\frac{a_0'a_5}{\omega}+\frac{a_0a_1'}{\omega}. \tag{134}$$

It may be noted that all terms present in the coefficients of the cubic equation (131), given by Eqs. (132)-(134), are lower than unity. The exact solutions of the cubic equation (131) are

$$s_1=-\frac{A}{3}-\frac{\sqrt[3]{2}\left(3B-A^2\right)}{3F}+\frac{F}{3\sqrt[3]{2}}, \tag{135}$$

$$s_2=-\frac{A}{3}+\frac{\left(1+i\sqrt{3}\right)\left(3B-A^2\right)}{2^{2/3}3F}-\frac{\left(1-i\sqrt{3}\right)F}{6\sqrt[3]{2}}, \tag{136}$$

$$s_3=-\frac{A}{3}+\frac{\left(1-i\sqrt{3}\right)\left(3B-A^2\right)}{2^{2/3}3F}-\frac{\left(1+i\sqrt{3}\right)F}{6\sqrt[3]{2}}, \tag{137}$$

where

$$F\equiv\sqrt[3]{-2A^3+9AB-27C+3\sqrt{3}\sqrt{\Delta}} \tag{138}$$

with the discriminant

$$\Delta\equiv -A^2B^2+4B^3+4A^3C-18ABC+27C^2. \tag{139}$$

The sign of Δ determines the nature of the roots (135)-(137); only one of the following three cases is possible: one real and two complex conjugate roots, if $\Delta<0$; three real and distinct roots, if $\Delta>0$ and three real roots, one different and two identical, if $\Delta=0$. Taking into account the orders of magnitude of small quantities (119), the explicit expressions of the three roots Eqs. (135)-(137), are given up to first order in the small quantities, i. e., up to k^2 order, as

$$s_{3,4} \simeq \frac{1}{2}\left(a_1 + a_3 - \frac{a_6}{\omega a_3}\right) \mp \frac{1}{2}\sqrt{\left(a_1 + a_3 - \frac{a_6}{\omega a_3}\right)^2 - 4a_1 a_3\left(1 - \frac{R}{R_c}\right)}, \tag{140}$$

$$s_5 \simeq a_5 + \frac{a_6}{\omega a_3}, \tag{141}$$

$$\frac{R}{R_c} \equiv -\left(\frac{a_0'}{\omega a_1 a_3} + \frac{a_0 a_1'}{\omega a_1^2 a_3} + \frac{a_0 a_1'}{\omega a_1 a_3^2}\right). \tag{142}$$

In Eq. (142) R is the Rayleigh number and R_c denotes its critical value. The radicand of (140) is the discriminant

$$\Delta \equiv \left(a_1 + a_3 - \frac{a_6}{\omega a_3}\right)^2 - 4a_1 a_3\left(1 - \frac{R}{R_c}\right). \tag{143}$$

It should be noted that according to the orders of magnitude of the quantities (119) contained in (140) and (141), from Eq. (143) it follows that $\Delta > 0$. Consequently, the roots (140)-(141) are real and distinct. Up to first order in the amounts (119), these roots are rewritten in terms of the variables λ_i as

$$\lambda_{3,4} \simeq \frac{1}{2}\left(D_T k^2 + \sigma_3 k^2 - \frac{\Omega^2 K_I k^4}{\rho_0 \sigma_3 k^2}\right) \mp \frac{1}{2}\sqrt{\left(D_T k^2 + \sigma_3 k^2 - \frac{\Omega^2 K_I k^4}{\rho_0 \sigma_3 k^2}\right)^2 - 4D_T k^2 \sigma_3 k^2\left(1 - \frac{R}{R_c}\right)}, \tag{144}$$

$$\lambda_5 \simeq \frac{K_I k^2}{\gamma_1} + \frac{\Omega^2 K_I k^4}{\rho_0 \sigma_3 k^2}, \tag{145}$$

where Eq. (142) have been rewritten in the form

$$\frac{R}{R_c} \equiv -\left[\frac{gX\beta\frac{k_\perp^2}{k^2}}{D_T k^2 \sigma_3 k^2} + \frac{g\alpha\beta\frac{k_\perp^2}{k^2}\Omega\chi_a k^2}{\left(D_T k^2\right)^2 \sigma_3 k^2} + \frac{g\alpha\beta\frac{k_\perp^2}{k^2}\Omega\chi_a k^2}{D_T k^2\left(\sigma_3 k^2\right)^2}\right]. \tag{146}$$

Equation (144) corresponds to a pair of visco-heat modes which result from the coupling between the thermal and shear modes. Their existence is entirely due to the presence of both, the uniform temperature gradient and the constant gravitational field, or only the gravity field. According to the orders of magnitude of the material proerties and experimental parameters indicated in Eq. (146), the first term is of order 10^{-15}, whereas the second and third terms are of 10^{-16}. Also, the discriminant Eq. (143) takes the form

$$\Delta \equiv \left(D_T k^2 + \sigma_3 k^2 - \frac{\Omega^2 K_I k^4}{\rho_0 \sigma_3 k^2}\right)^2 - 4D_T k^2 \sigma_3 k^2\left(1 - \frac{R}{R_c}\right). \tag{147}$$

In Eq. (146) the presence of χ_a in the second and third terms is indicative that the system under study is a nematic; besides, such term is an order of magnitude greater than the first. If in this same expression $\chi_a = 0$, which occurs in the isotropic limit, then

$$\frac{R}{R_c} \equiv -\frac{gX\beta}{D_T k^2 \sigma_3 k^2}\frac{k_\perp^2}{k^2}, \tag{148}$$

which has the same structure of the corresponding expression reported for a simple fluid [9], [22], [25]. This results allow us to quantify the effect produced in the modes, by α and g. Their influence is due to the coupling of the small quantities present in the factor $g\alpha\beta\frac{k_\perp^2}{k^2}$. Also, it is worth noting that the presence of α and g produces a coupling between the thermal and shear difusive modes, as may well be seen in Eq. (144).

3.2 Values of R

3.2.1 Critical value $(R = R_c)$

Some special values of R are of particular interest. For instance, if R reaches its critical value R_c, then $\Delta = \left(D_T k^2 + \sigma_3 k^2 - \frac{\Omega^2 K_I k^4}{\rho_0 \sigma_3 k^2}\right)^2$ and hence the modes (144) and (145) are simplified in the form

$$\lambda_3 \simeq 0, \tag{149}$$

$$\lambda_4 \simeq D_T k^2 + \sigma_3 k^2 - \frac{\Omega^2 K_I k^4}{\rho_0 \sigma_3 k^2}, \tag{150}$$

$$\lambda_5 \simeq \frac{K_I k^2}{\gamma_1} + \frac{\Omega^2 K_I k^4}{\rho_0 \sigma_3 k^2}, \tag{151}$$

which are in agreement with those reported in literature in this limit [23], [24]. In this situation λ_3 vanishes, λ_5 is virtually unchanged, while λ_4 has contributions from the thermal and shear difusive modes. It should be pointed out that this phenomenon also occurs in the simple fluid, where there are two diffusive modes, one of them also vanishes, and the other one has contributions from the shear and thermal modes [9], [25]. For a simple fluid, these features have been predicted theoretically, and even more, corroborated experimentally. These results suggest that it might be feasible to verify them experimentally also for nematics. It should be stressed that the results obtained in this limit do not coincide with those reported for a NLC, according to which the director mode tends to zero, the shear mode does not change and there is an additional mode which is the sum of the thermal and director modes [23], [24].

3.2.2 Equilibrium state $(R = 0)$

In the absence of temperature the gradient α and the gravitational field g, $R = 0$ and from Eqs. (144) and (145) the corresponding expressions for the thermal, shear and director diffusive modes in the equilibrium state (identified by the superscript e) are readily obtained. The correponding expressions reduce to

$$\lambda_3^e \simeq D_T k^2, \tag{152}$$

$$\lambda_4^e \simeq \sigma_3 k^2 - \frac{\Omega^2 K_I k^4}{\rho_0 \sigma_3 k^2}, \tag{153}$$

$$\lambda_5^e \simeq \frac{K_I k^2}{\gamma_1} + \frac{\Omega^2 K_I k^4}{\rho_0 \sigma_3 k^2}, \tag{154}$$

which are well known results in the literature [26], [27], [28]. In this homogeneous thermodynamic equilibrium state, the decay rates Eqs. (152)-(154) are purely diffusive.

3.2.3 Visco-heat propagation modes

It has already been mentioned that owing to the orders of magnitude of the small quantities (119), the roots (140) and (141) are real and different. Nevertheless, it may happen that these roots may be transformed into one real and two complex conjugate roots. This occurs if $\Delta < 0$ in Eq. (140) and if

$$\frac{R}{R_c} < -\frac{\left(-D_T k^2 + \sigma_3 k^2 - \frac{\Omega^2 K_I k^4}{\rho_0 \sigma_3 k^2}\right)^2}{4 D_T k^2 \sigma_3 k^2}. \tag{155}$$

If we consider the orders of magnitude of the involved quantities $D_T k^2 \sim 10^7$, $\sigma_3 k^2 \sim 10^8$ and $\frac{\Omega^2 K_I k^4}{\rho_0} \sim 10^{14}$, then in Eq. (155) $R/R_c \lesssim -10^1$ is negative. Thereby, Eq. (155) implies that there are two visco-heat propagating modes when $\frac{R}{R_c} < 0$ and $R/R_c \lesssim -10^1$. According to Eq. (146), this occurs if α changes its sign and increases by several orders of magnitude, situation that may be achieved by reversing the direction in which the temperature gradient is applied, i. e., when heating from below, and by increasing its intensity. To our knowledge, there are no theoretical analysis nor exprimental evidence for the existence of visco-heat propagating modes in nematic liquid crystals under the presence of a temperature gradient and an uniform gravitational field. Given that in simple fluids, under these conditions, there are analytical [25] and experimental [40] studies that supports the presence of visco-heat propagation modes, this prediction suggests that it may be worth to design experiments to corroborate this phenomenon in nematics.

3.3 Transverse modes

The roots of the quadratic polynomial $p_T(\lambda)$ of the matrix N^T given by Eq. (96), are the nematic transverse modes. According to Eqs. (93), (94), (99), Eq. (90) may be written as

$$\frac{\partial}{\partial t} \overrightarrow{Z}^T(\overrightarrow{k}, t) = -N^T \overrightarrow{Z}^T(\overrightarrow{k}, t) + \overrightarrow{\Xi}^T(\overrightarrow{k}, t), \tag{156}$$

which it is the linear stochastic equation for the transverse variables.

3.3.1 Shear and director transverse modes

According to Eq. (96) the shear and director transverse modes are the roots of

$$\lambda^2 - \left(\sigma_4 k^2 + \frac{K_{II} k^2}{\gamma_1}\right)\lambda + \sigma_4 k^2 \frac{K_{II} k^2}{\gamma_1} + \frac{\lambda_+^2 K_{II} k^2 k_z^2}{\rho_0} = 0. \tag{157}$$

Following again the approximate method of small quantities used previously, the small quantities quantities $\sigma_4 k^2$, $K_{II}k^2/\gamma_1$ and $\lambda_+^2 K_{II} k^2 k_z^2/\rho_0$, may be identifid in Eq. (157). We define the small or reduced quantities

$$a_4 \equiv \frac{\sigma_4 k^2}{\omega}, \quad a_5' \equiv \frac{K_{II}k^2}{\gamma_1 \omega}, \quad a_6' \equiv \frac{\lambda_+^2 K_{II}}{\rho_0 \omega} k^2 k_z^2. \tag{158}$$

Since for typical nematics λ_1 is of the order of unity, that $\gamma_1 \sim 10^{-1}$, $\sigma_4 \sim 10^{-2}$, $K_{II} \sim 10^{-6}$ [29], and also taking into account that $c_s \sim 10^5$, $k \sim 10^5$, $g \sim 10^3$, the quantities in Eq. (158) have the orders of magnitude $a_4 \sim 10^{-2}$, $a_5' \sim 10^{-5}$ and $a_6' \sim 10^4$. Therefore, in terms of the reduced variable $s \equiv \lambda/\omega$, Eq. (157) takes the form

$$s^2 + A'' s + B'' = 0, \tag{159}$$

with

$$A'' \equiv -a_4 - a_5', \tag{160}$$

$$B'' \equiv a_4 a_5' + \frac{a_6'}{\omega}. \tag{161}$$

The analytic solutions of Eq. (159) are

$$s_+ = -\frac{1}{2}A'' + \frac{1}{2}\sqrt{\Delta''}, \tag{162}$$

$$s_- = -\frac{1}{2}A'' - \frac{1}{2}\sqrt{\Delta''}, \tag{163}$$

in which the discriminant is given by

$$\Delta'' \equiv A''^2 - 4B''. \tag{164}$$

According to the orders of magnitude of the quantities (158), the discriminant Eq. (164) may be simplified to $\Delta' \simeq a_4^2 - 2a_4 a_5' > 0$, which implies that $\Delta'' > 0$ always. Consequently, the solutions (162) and (163) will be real and different, namely,

$$s_+ \simeq a_4 - \frac{a_6'}{\omega a_4}, \tag{165}$$

$$s_- \simeq a_5' + \frac{a_6'}{\omega a_4}. \tag{166}$$

As before, they are rewritten as

$$\lambda_6 = \sigma_4 k^2 - \frac{\lambda_+^2 K_{II} k^2 k_z^2}{\rho_0 \sigma_4 k^2}, \tag{167}$$

$$\lambda_7 = \frac{K_{II}k^2}{\gamma_1} + \frac{\lambda_+^2 K_{II} k^2 k_z^2}{\rho_0 \sigma_4 k^2}. \tag{168}$$

It should be noted that the shear and director diffusive transverse modes found previously, Eqs. (167) and (168), completely match with those already reported for nematic systems [26], [27], [29].

4 Discussion and conclusions

The theoretical results obtained in this work indicate that the presence of a thermal gradient α and gravitational field g produced its most significant effect only on the visco-heat $\lambda_{3,4}$ (formed by the coupling of the shear an thermal modes) and director λ_5 longitudinal modes. In these modes the effect is of the order of 10^{-9}. In contrast, in the other two remaining sound propagating longitudinal modes, λ_1 and λ_2, g is the only external force that produces a small influence of the order of 10^{-24}. In contrast, the shear λ_6 and director λ_7 transverse modes are not affected by these external forces.

The analytical expressions found for the nematodynamic modes are more general than the previously reported in literature, but when $R = 0$, they reduce to the corresponding expressions already reported for a nematic in the equilibrium state. Also, in the isotropic limit, these modes reduced to those of a simple fluid.

When R reaches its critical value R_c, $R = R_c$, λ_3 vanishes, λ_5 is virtually unchanged, while λ_4 has contributions from the thermal and shear difusive modes. It should be remaked that this behavior also occurs for a simple fluid.In this case there are two diffusive modes, one of them also vanishes, and the other one has contributions from the shear and thermal modes [9], [25]. For a simple fluid, these features have been predicted theoretically, and even more, verified experimentally. These results suggest that it might be feasible to verify them experimentally for nematics as well. Our results obtained in this limit do not coincide with those reported for a NLC [23], [24], according to which the director mode tends to zero, the shear mode does not change and there is an additional mode which is the sum of the thermal and director modes .

If we consider the orders of magnitude of the involved quantities $D_T k^2 \sim 10^7$, $\sigma_3 k^2 \sim 10^8$ and $\frac{\Omega^2 K_I k^4}{\rho_0} \sim 10^{14}$, then from Eq. (155) $R/R_c \lesssim -10^1$. Thereby, Eq. (155) implies that there are two visco-heat *propagating* modes when $\frac{R}{R_c} < 0$ and $R/R_c \lesssim -10^1$, a prediction which is not contained in Refs. [23], [24], and is valid for a simple fluid [25], [40]. Since the existence of these propagative modes has only been predicted and verified experimentally in simple fluids, our prediction for NLC suggests that their existence might be also verified experimentally.

In the literature, the nematic longitudinal hydrodynamic modes in a steady state have been studied in Refs. [23], [24] for the same $NESS$ considered in this work. These works predict that the thermal and director diffusive modes are coupled. We believe that this result is not correct, because in the isotropic limit, these modes do not reduce to the corresponding visco-heat modes of a simple fluid [25]. In contrast, the analytical expressions that we have found for these nematodynamic modes imply that the heat and shear modes of the NLC are coupled and do reduce to those of simple fluid in the isotropic limit.

Appendix A

The sums of stochastic noises ζ_m (whith $m = 1 \ldots 6$) in Eqs. (98) and (99) are defined as

$$\zeta_1\left(\overrightarrow{k}, t\right) \equiv -i\left(\frac{\gamma - 1}{\rho_0 T_0 c_p}\right)^{1/2} k_j \widetilde{\pi}_j, \tag{169}$$

$$\zeta_2\left(\overrightarrow{k}, t\right) \equiv -\frac{k_j}{\rho_0^{1/2} k}\left(k_x \widetilde{\Sigma}_{xj} + k_y \widetilde{\Sigma}_{yj} + k_z \widetilde{\Sigma}_{zj}\right), \tag{170}$$

$$\zeta_3\left(\overrightarrow{k}, t\right) \equiv -i\left(\frac{1}{\rho_0 T_0 c_p}\right)^{1/2} k_j \widetilde{\pi}_j, \tag{171}$$

$$\zeta_4\left(\overrightarrow{k}, t\right) \equiv -\frac{ik_z}{\rho_0^{1/2}}\frac{k_j}{k^2}\left(k_x \widetilde{\Sigma}_{xj} + k_y \widetilde{\Sigma}_{yj} + k_z \widetilde{\Sigma}_{zj}\right) + \frac{ik_j}{\rho_0^{1/2}}\widetilde{\Sigma}_{zj}, \tag{172}$$

$$\zeta_5\left(\overrightarrow{k}, t\right) \equiv i\rho_0^{1/2} c_s \frac{k_x}{k}\widetilde{\Upsilon}_x + i\rho_0^{1/2} c_s \frac{k_y}{k}\widetilde{\Upsilon}_y, \tag{173}$$

$$\zeta_6\left(\overrightarrow{k}, t\right) \equiv -\frac{k_j}{\rho_0^{1/2} k}\left(k_x \widetilde{\Sigma}_{yj} - k_y \widetilde{\Sigma}_{xj}\right), \tag{174}$$

$$\zeta_7\left(\overrightarrow{k}, t\right) \equiv i\rho_0^{1/2} c_s \frac{k_x}{k}\widetilde{\Upsilon}_y - i\rho_0^{1/2} c_s \frac{k_y}{k}\widetilde{\Upsilon}_x, \tag{175}$$

where $j = x, y, z$.

The autocorrelations and cross-correlations of the stochastic noises at the two different points $\left(\overrightarrow{k}, \omega\right)$ and $(\overrightarrow{q}, w)$, are calculated by using the fluctuation-dissipation relations Eqs. (33)-(35) averaged over the steady state. They are given by

$$\begin{aligned}\left\langle \zeta_1\left(\overrightarrow{k}, \omega\right)\zeta_1^*\left(\overrightarrow{q}, w\right)\right\rangle^{st} = \frac{2k_B \widetilde{T}^{st}\left(\overrightarrow{k}, \overrightarrow{q}, \overrightarrow{s}\right)(\gamma - 1)}{\rho_0 c_p}\,&[\kappa_\perp (k_x q_x + k_y q_y) \\ &+\kappa_\parallel k_z q_z \delta(\omega - w)\big],\end{aligned} \tag{176}$$

$$\begin{aligned}\left\langle \zeta_2\left(\overrightarrow{k}, \omega\right)\zeta_2^*\left(\overrightarrow{q}, w\right)\right\rangle^{st} &= \frac{2k_B \widetilde{T}^{st}\left(\overrightarrow{k}, \overrightarrow{q}, \overrightarrow{s}\right)}{\rho_0 k q}[(\nu_2 + \nu_4)\left(k_x^2 q_x^2 + k_y^2 q_y^2\right) \\ &+ (\nu_4 - \nu_2)\left(k_y^2 q_x^2 + k_x^2 q_y^2\right) + 4\nu_2 k_x k_y q_y q_x + 4\nu_3 (k_x q_x + k_y q_y) k_z q_z \\ &+\nu_5\left(q_x^2 + q_y^2\right)k_z^2 + \nu_5\left(k_x^2 + k_y^2\right)q_z^2 \\ &+ (2\nu_1 + \nu_2 - \nu_4 + 2\nu_5) k_z^2 q_z^2]\delta(\omega - w),\end{aligned} \tag{177}$$

$$\begin{aligned}\left\langle \zeta_3\left(\overrightarrow{k}, \omega\right)\zeta_3^*\left(\overrightarrow{q}, w\right)\right\rangle^{st} = \frac{2k_B \widetilde{T}^{st}\left(\overrightarrow{k}, \overrightarrow{q}, \overrightarrow{s}\right)}{\rho_0 c_p}\,&[\kappa_\perp (k_x q_x + k_y q_y) \\ &+\kappa_\parallel k_z q_z\big]\,\delta(\omega - w),\end{aligned} \tag{178}$$

$$\left\langle \zeta_4\left(\overrightarrow{k},\omega\right)\zeta_4^*\left(\overrightarrow{q},w\right)\right\rangle^{st} = \frac{2k_B\widetilde{T}^{st}\left(\overrightarrow{k},\overrightarrow{q},\overrightarrow{s}\right)k_zq_z}{\rho_0k^2q^2}\left[(\nu_2+\nu_4)\left(k_x^2q_x^2+k_y^2q_y^2\right)\right.$$
$$+(\nu_4-\nu_2)\left(k_y^2q_x^2+k_x^2q_y^2\right)+4\nu_2k_xk_yq_yq_x+4\nu_3\left(k_xq_x+k_yq_y\right)k_zq_z$$
$$+\nu_5\left(q_x^2+q_y^2\right)k_z^2+\nu_5\left(k_x^2+k_y^2\right)q_z^2$$
$$\left.+(2\nu_1+\nu_2-\nu_4+2\nu_5)k_z^2q_z^2]\delta(\omega-w)\right], \tag{179}$$

$$\left\langle \zeta_5\left(\overrightarrow{k},\omega\right)\zeta_5^*\left(\overrightarrow{q},w\right)\right\rangle^{st} = \frac{2k_B\widetilde{T}^{st}\left(\overrightarrow{k},\overrightarrow{q},\overrightarrow{s}\right)\rho_0c_s^2}{\gamma_1kq}\left(k_xq_x+k_yq_y\right)\delta(\omega-w), \tag{180}$$

$$\left\langle \zeta_6\left(\overrightarrow{k},\omega\right)\zeta_6^*\left(\overrightarrow{q},w\right)\right\rangle^{st} = \frac{2k_B\widetilde{T}^{st}\left(\overrightarrow{k},\overrightarrow{q},\overrightarrow{s}\right)}{\rho_0kq}\left[\nu_2\left(k_xq_x+k_yq_y\right)^2\right.$$
$$\left.-\nu_2\left(k_yq_x-k_xq_y\right)^2+\nu_3\left(k_xq_x+k_yq_y\right)k_zq_z\right]\delta(\omega-w), \tag{181}$$

$$\left\langle \zeta_7\left(\overrightarrow{k},\omega\right)\zeta_7^*\left(\overrightarrow{q},w\right)\right\rangle^{st} = \frac{2k_B\widetilde{T}^{st}\left(\overrightarrow{k},\overrightarrow{q},\overrightarrow{s}\right)\rho_0c_s^2}{\gamma_1kq}\left(k_xq_x+k_yq_y\right)\delta(\omega-w); \tag{182}$$

and by

$$\left\langle \zeta_1\left(\overrightarrow{k},\omega\right)\zeta_3^*\left(\overrightarrow{q},w\right)\right\rangle^{st} = \frac{2k_B\widetilde{T}^{st}\left(\overrightarrow{k},\overrightarrow{q},\overrightarrow{s}\right)(\gamma-1)^{1/2}}{\rho_0c_p} \tag{183}$$
$$\left[\kappa_\perp\left(k_xq_x+k_yq_y\right)+\kappa_\parallel k_zq_z\right]\delta(\omega-w),$$

$$\left\langle \zeta_3\left(\overrightarrow{k},\omega\right)\zeta_1^*\left(\overrightarrow{q},w\right)\right\rangle^{st} = \frac{2k_B\widetilde{T}^{st}\left(\overrightarrow{k},\overrightarrow{q},\overrightarrow{s}\right)(\gamma-1)^{1/2}}{\rho_0c_p} \tag{184}$$
$$\left[\kappa_\perp\left(k_xq_x+k_yq_y\right)+\kappa_\parallel k_zq_z\right]\delta(\omega-w),$$

$$\left\langle \zeta_2\left(\overrightarrow{k},\omega\right)\zeta_4^*\left(\overrightarrow{q},w\right)\right\rangle^{st} = -\frac{2ik_B\widetilde{T}^{st}\left(\overrightarrow{k},\overrightarrow{q},\overrightarrow{s}\right)q_z}{\rho_0kq^2}\left[(\nu_2+\nu_4)\left(k_x^2q_x^2+k_y^2q_y^2\right)\right.$$
$$+(\nu_4-\nu_2)\left(k_y^2q_x^2+k_x^2q_y^2\right)+4\nu_2k_xk_yq_yq_x+4\nu_3\left(k_xq_x+k_yq_y\right)k_zq_z$$
$$+\nu_5\left(q_x^2+q_y^2\right)k_z^2+\nu_5\left(k_x^2+k_y^2\right)q_z^2$$
$$+(2\nu_1+\nu_2-\nu_4+2\nu_5)k_z^2q_z^2]\delta(\omega-w), \tag{185}$$

$$\left\langle \zeta_4\left(\overrightarrow{k},\omega\right)\zeta_2^*\left(\overrightarrow{q},w\right)\right\rangle^{st} = \frac{2ik_B\widetilde{T}^{st}\left(\overrightarrow{k},\overrightarrow{q},\overrightarrow{s}\right)k_z}{\rho_0 k^2 q}[(\nu_2+\nu_4)\left(k_x^2q_x^2+k_y^2q_y^2\right)$$
$$+(\nu_4-\nu_2)\left(k_y^2q_x^2+k_x^2q_y^2\right)+4\nu_2k_xk_yq_yq_x+4\nu_3\left(k_xq_x+k_yq_y\right)k_zq_z$$
$$+\nu_5\left(q_x^2+q_y^2\right)k_z^2+\nu_5\left(k_x^2+k_y^2\right)q_z^2$$
$$+(2\nu_1+\nu_2-\nu_4+2\nu_5)\,k_z^2q_z^2]\delta\left(\omega-w\right). \tag{186}$$

In Eqs. (176)-(186), $\widetilde{T}^{st}\left(\overrightarrow{k},\overrightarrow{q},\overrightarrow{s}\right)$ can be identified as the spatial Fourier transform (66) of (3),

$$\widetilde{T}^{st}(\overrightarrow{k},\overrightarrow{q},\overrightarrow{s}) \equiv T_0\delta\left(\overrightarrow{k}-\overrightarrow{q}\right) \tag{187}$$
$$-\frac{\alpha}{2is}\left[\delta\left(\overrightarrow{k}-\overrightarrow{q}-\overrightarrow{s}\right)-\delta\left(\overrightarrow{k}-\overrightarrow{q}+\overrightarrow{s}\right)\right].$$

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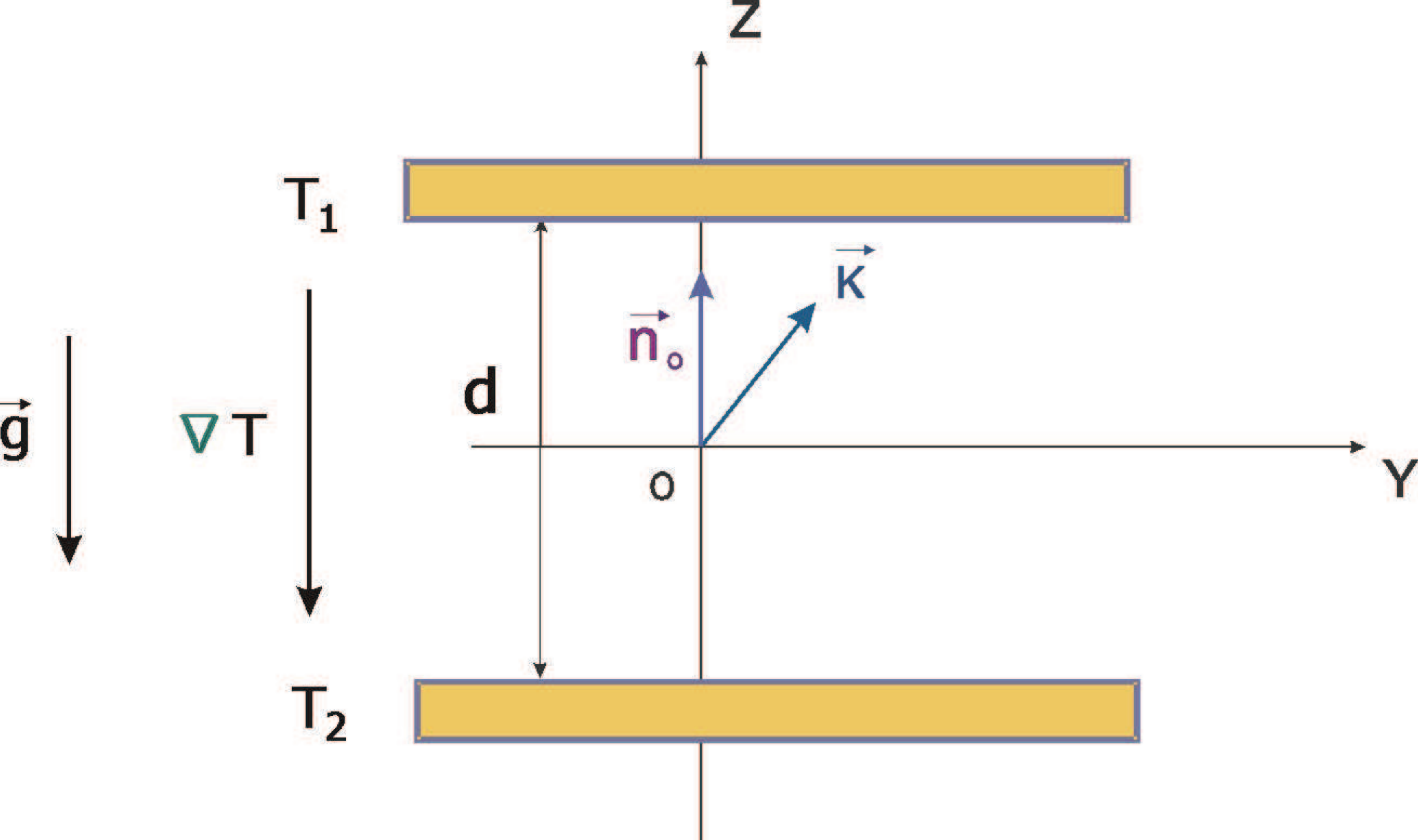

Z
T_1
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